1. (10 points) If $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite show that

$$|a_{i,j}| \leq \frac{1}{2}\left(a_{i,i} + a_{j,j}\right)$$

holds for all $1 \leq i, j \leq n$.

2. The Gerschgorin Disk Theorem states that for any $n \times n$ matrix A, every eigenvalue of A lies in at least one of the n circular disks in the complex plane with centers a_{ii} and radii $\Sigma_{j \neq i} |a_{ij}|$.

   (a) (5 points) Prove the Gerschgorin Disk Theorem.

   (b) (5 points) Use the Gerschgorin Disk Theorem to show that all of the eigenvalues of

$$A = \begin{bmatrix} 1 & 10000 \\ 0 & 1 \end{bmatrix}$$

   lie on a disk, $\{\lambda : |\lambda - 1| \leq 10^4\}$. Do not solve the eigenvalue problem.

   (c) (10 points) Find a way to show that the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 10000 \\ 0 & 1 \end{bmatrix}$$

   actually lie on a much smaller disk, $\{\lambda : |\lambda - 1| \leq 10^{-4}\}$. Once again, do not solve the eigenvalue problem.

3. Suppose that $A \in \mathbb{C}^{m \times n}$, $m \geq n$, has the block form

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where $A_1 \in \mathbb{C}^{n \times n}$ is invertible, and $A_2 \in \mathbb{C}^{(m-n) \times n}$ is arbitrary. Let $\sigma_1(A_1)$, $\sigma_1(A_2)$, and $\sigma_1(A)$ be the largest singular values of A_1, A_2, and A, respectively. Similarly, let $\sigma_{\min}(A_1)$, $\sigma_{\min}(A_2)$, and $\sigma_{\min}(A)$ be the smallest singular values of A_1, A_2, and A, respectively. Show that:

(a) (10 points) $\sigma_1(A_1) + \sigma_1(A_2) \geq \sigma_1(A) \geq \max\{\sigma_1(A_1), \sigma_1(A_2)\}$.

(b) (10 points) $\sigma_{\min}(A_1) + \sigma_1(A_2) \geq \sigma_{\min}(A) \geq \max\{\sigma_{\min}(A_1), \sigma_{\min}(A_2)\} > 0$.

4. Let

$$B = \begin{pmatrix} 3 & -6 & 41/5 \\ 0 & 1 & 1 \\ 4 & -8 & 63/5 \end{pmatrix}.$$

(a) (10 points) Find the QR decomposition of B using Gram-Schmidt. *Hint: Every entry in R is an integer!*

(b) (10 points) Write down the unitary Householder reflector matrix that you should multiply against B in order to zero out all but the first entry of its first column.

5. (10 points) Let $T \in C^{n\times n}$ be an upper triangular matrix, and let $\epsilon > 0$. Show that there is a consistent matrix norm $\|\cdot\| : C^{n\times n} \to \mathcal{R}$ depending on T and ϵ such that

$$\|T\| \leq \rho(T) + \epsilon,$$

where $\rho(T)$ is the spectral radius of T.

6. (10 points) Suppose $A = (a_{i,j}) \in \mathcal{R}^{n \times m}$, $n \geq m$, $\text{rank}(A) = m$, and $A = QR$, where $Q \in \mathcal{R}^{n \times n}$ is orthogonal and $R \in \mathcal{R}^{n \times m}$ is upper triangular. Let

$$\tilde{A} = \begin{bmatrix} A_1 \\ z^T \\ A_2 \end{bmatrix}$$

where $\tilde{A} \in \mathcal{R}^{(n+1) \times m}$, and

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

What is the QR factorization of $\tilde{A}$, $\tilde{A} = \tilde{Q}\tilde{R}$, in terms of Q and R, where $\tilde{Q} \in \mathcal{R}^{(n+1) \times (n+1)}$ is orthogonal and $\tilde{R} \in \mathcal{R}^{(n+1) \times m}$ is upper triangular?

7. (10 points) Prove that if $A \in \mathcal{R}^{n \times n}$ is of rank r, then it depends upon $r(2n - r)$ degrees of freedom.

1. (20 points) Let the vector field $f\colon R^m \to R^m$ be divergence free,

$$\nabla \cdot f(y) = \frac{\partial f_1}{\partial y_1} + \cdots + \frac{\partial f_m}{\partial y_m} = 0.$$

Show that the following autonomous ODE system preserves volume:

$$y' = f(y), \qquad 0 \le t \le b, \tag{1}$$
$$y(0) = y_0, \tag{2}$$

in the sense that if $B(0)$ is a volume in R^m, then the set $B(t)$ generated by the evolution of the set $B(0)$ under the flow defined by equations (1) and (2) preserves volume:

$$\text{Volume}(B(t)) = \text{Volume}(B(0)),$$

where $B(t) = \{y(t; y_0) : y_0 \in B(0)\}$.

2. (20 points) Define

$$\left(\begin{array}{c} s \\ j \end{array} \right) = \frac{s(s-1)\cdots(s-j+1)}{j!}, \quad \left(\begin{array}{c} s \\ 0 \end{array} \right) = 1,$$

where j is an integer and s is real. Let

$$\gamma_j = (-1)^j \int_0^1 \left(\begin{array}{c} -s \\ j \end{array} \right) ds.$$

Prove that for $m = 0, 1, 2, \ldots$, the following recursive formula for γ_m holds,

$$\gamma_m + \frac{1}{2}\gamma_{m-1} + \frac{1}{3}\gamma_{m-2} + \cdots + \frac{1}{m+1}\gamma_0 = 1,$$

where $\gamma_0 = 1$ for $m = 0$, $\gamma_1 = 1 - \frac{1}{2}\gamma_0$ for $m = 1$, and so on.

3. (10 points) Consider numerical methods for an ODE initial value problem.
   (a) (2 points) State the definition of consistency of numerical methods.
   (b) (2 points) State the definition of zero-stability of numerical methods.
   (c) (2 points) State the definition of convergency of numerical methods.
   (d) (4 points) Prove that consistency plus zero-stability implies convergence.

4. (a) (5 points) Show that the implicit trapezoidal method is zero-stable for the ODE initial value problem

$$y' = f(t,y), \qquad 0 \le t \le b, \tag{3}$$
$$y(0) = c, \tag{4}$$

where f is assumed to be sufficiently smooth and bounded so that the unique existence of a solution is guaranteed with as many bounded derivatives as needed.

(b) (5 points) Prove that the implicit trapezoidal method is convergent of second-order accuracy.

5. (a) (5 points) State the definition of A-stability of a numerical method for an ODE initial value problem.

   (b) (5 points) Show that the backward Euler method is A-stable for

$$y' = f(t,y) \qquad 0 \le t \le b, \tag{5}$$
$$y(0) = c. \tag{6}$$

.

6. Consider a two-step backward differentiation formula (BDF)

$$y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} = h\beta_0 f_n,$$

where $\beta_0 \neq 0$.

(a) (5 points) Determine the unknown coefficients α_1, α_2 and β_0 so that the scheme is second-order accurate.

(b) (5 points) Show that the above method is indeed of second-order accuracy by computing the local truncation error.

7. (5 points) Use the algebraic characterization of stability of BDFs to show that applying the BDF
$$y_n = y_{n-2} + \frac{1}{3}h(f_n + f_{n-1} + f_{n-2})$$
to $y' = \lambda y$ is unstable when $\lambda < 0$.

8. Consider the family of linear multistep methods

$$y_n = \alpha y_{n-1} + \frac{h}{2}(2(1-\alpha)f_n + 3\alpha f_{n-1} - \alpha f_{n-2}),$$

where α is a real parameter.

(a) (5 points) Analyze consistency and order of the methods as functions of α, determining the value α^* for which the resulting method has maximal order.

(b) (5 points) Study the zero-stability of the method with $\alpha = \alpha^*$.

9. (5 points) Formulate the multiple shooting method for the linear problem

$$\mathbf{y}'(t) = A(t)\mathbf{y}(t) + \mathbf{q}(t), \quad 0 \le t \le b, \tag{7}$$
$$B_0\mathbf{y}(0) + B_b\mathbf{y}(b) = \mathbf{b}, \tag{8}$$

where $\mathbf{y}$, $\mathbf{q}$ and $\mathbf{b}$ have m components, and $A(t)$, B_0, and B_b are $m \times m$ matrices.